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FAST TRACK COMMUNICATION

Natural superoscillations in monochromatic waves in D dimensions

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Abstract

For monochromatic waves satisfying the Helmholtz equation with wavenumber k_0, superoscillations correspond to local wavenumbers (magnitude of phase gradient) greater than k_0. Large values of local wavenumber are associated with phase singularities. For isotropic random waves (superpositions of many nonevanescent waves) in D dimensions, we show that the probability that a point in the field is superoscillatory increases from 0.293 to 0.394 as D increases from 1 to infinity. The peculiar case $D = 1$ is examined in detail.

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1. Introduction

It is now understood that band-limited functions can oscillate faster than their fastest Fourier components [1, 2]. This apparently paradoxical behaviour is demystified by the observation, now well understood [3], that such 'superoscillations' occur in regions where the amplitude of the function is small. Several constructions have been devised, leading to functions that superoscillate arbitrarily rapidly in arbitrarily large domains.

Here we will consider superoscillations occurring naturally in band-limited functions of a special but physically important type: monochromatic complex scalar waves in D dimensions. These have the form

$$\psi(\boldsymbol{r}) = u(\boldsymbol{r}) + \mathrm{i}v(\boldsymbol{r}) = \rho(\boldsymbol{r}) \exp\{\mathrm{i}\chi(\boldsymbol{r})\}, \qquad \boldsymbol{r} = \{x_1, \ldots, x_D\}, \tag{1.1}$$

with ψ satisfying the Helmholtz equation

$$\nabla^2\psi + k_0^2\psi = 0, \tag{1.2}$$

in which k_0 is the free-space wavenumber. Superoscillations will correspond to ψ varying on subwavelength scales, that is, scales smaller than the wavelength $2\pi/k_0$.

A natural measure of the oscillation at $\boldsymbol{r}$ is the rate at which the phase $\chi(\boldsymbol{r})$ is changing; this is the local wavenumber:

$$k(\boldsymbol{r}) \equiv |\nabla\chi(\boldsymbol{r})| = \mathrm{Im}[\nabla \log \psi(\boldsymbol{r})] = \frac{|u(\boldsymbol{r})\nabla v(\boldsymbol{r}) - v(\boldsymbol{r})\nabla u(\boldsymbol{r})|}{\rho(\boldsymbol{r})^2}. \tag{1.3}$$

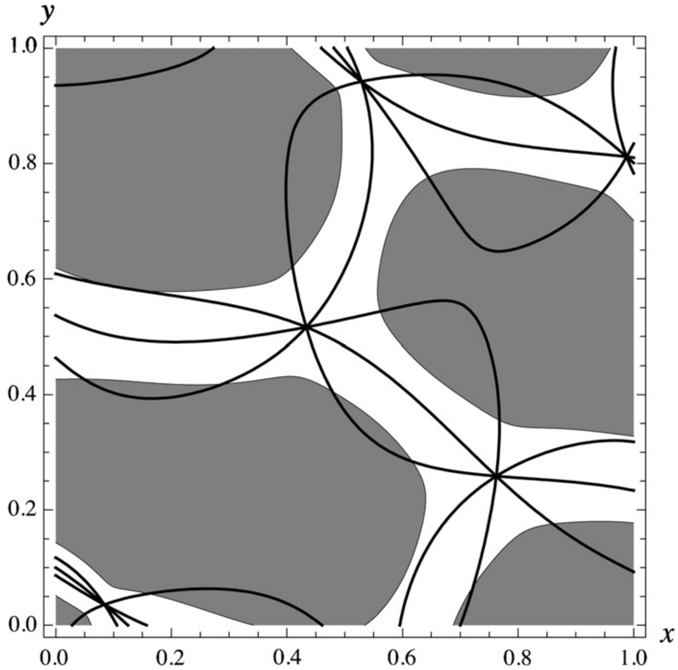

Figure 1. Superoscillations for $D = 2$, calculated from the plane-wave superposition (1.5) for $N = 10$ and $k_0 = 2\pi$ (i.e. wavelength $\lambda = 1$), showing wavefronts at phase intervals $\pi/2$, intersecting at the vortices (black lines) and the superocillatory region where $k(\boldsymbol{r}) > k_0$ (white).

There are two situations in which $k(\boldsymbol{r})$ can exceed k_0. In the first, the decomposition of ψ into plane waves includes evanescent waves. A two-dimensional example is

$$\psi(\boldsymbol{r}) = \exp\{\mathrm{i}Kx\}\exp\left\{-y\sqrt{K^2 - k_0^2}\right\}, \qquad K > k_0. \tag{1.4}$$

Obviously, $k(\boldsymbol{r}) = K$: there are subwavelength oscillations transverse to the decaying direction; these are not superoscillations, because the fast variation is present in the spectrum of ψ. We are interested in the contrary case, where ψ contains only real plane waves; then regions where $k(\boldsymbol{r}) > k_0$ correspond to genuine superoscillations. We expect these regions to be centred on wave vortices [4–7], that is, the phase singularities, where the functions vanish and the phase varies infinitely rapidly. By constructing waves with a cluster of many close-lying vortices, the superoscillations can be made faster and more numerous [7].

We seek to quantify the degree of superoscillation not in artificial constructions but in naturally occurring monochromatic waves. We model these by isotropic superpositions of many plane waves:

$$\psi(\boldsymbol{r}) = \sum_{n=1}^{N} a_n \exp\{\mathrm{i}\boldsymbol{k}_n \cdot \boldsymbol{r}\}, \qquad N \gg 1, \quad |\boldsymbol{k}_n| = k_0, \tag{1.5}$$

in which the amplitudes a_n are random complex numbers and the wavevector directions $\boldsymbol{k}_n/k_0$ are uniformly distributed. Figures 1 and 2 depict sample functions for $D = 2$ and $D = 3$. As expected, the regions of superoscillation (highlighted) include the wave vortices.

The simplest measure of superoscillation is the probability $P_{D\,\mathrm{super}}$ that $k(\boldsymbol{r}) > k_0$ for a randomly chosen point $\boldsymbol{r}$. We will calculate this for general D in section 2, using the fact

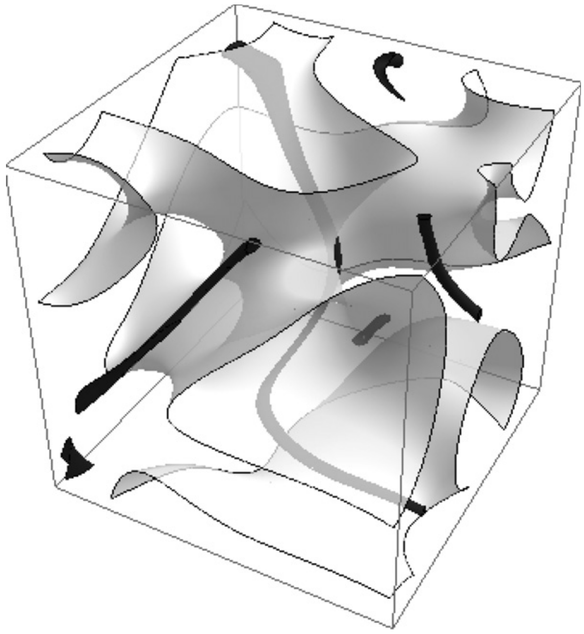

Figure 2. Superoscillations for $D = 3$, for a superposition (1.5) with $N = 100$. The shaded surface is the boundary $k(\boldsymbol{r}) = k_0$ of the superoscillatory region, which includes the phase singularities (black lines). The volume displayed is one cubic wavelength.

(a consequence of the central limit theorem) that the waves defined by (1.5) are isotropic Gaussian random functions. This generalizes the recent result $P_{2\,\mathrm{super}} = 1/3$ by one of us [8]. The case $D = 1$ is peculiar, and deserves a special discussion (section 3).

2. Superoscillation probability

The desired quantity is

$$P_{D\,\mathrm{super}} = \int_{k_0}^{\infty} \mathrm{d}k\, P_D(k), \tag{2.1}$$

where $P_D(k)$ is the probability distribution of $k(\boldsymbol{r})$ for waves in D dimensions. From (1.3), this is

$$\begin{aligned} P_D(k) &= \Omega_D k^{D-1} \left\langle \delta\left(\boldsymbol{k} - \frac{u\nabla v - v\nabla u}{\rho^2}\right)\right\rangle \\ &= \frac{\Omega_D k^{D-1}}{(2\pi)^D} \int \mathrm{d}^D \boldsymbol{s} \exp\{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{s}\} \left\langle \exp\left\{\mathrm{i}\boldsymbol{s}\cdot \frac{u\nabla v - v\nabla u}{\rho^2}\right\}\right\rangle, \end{aligned} \tag{2.2}$$

where

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma\left(\frac{1}{2}D\right)} \tag{2.3}$$

is the surface area of the unit sphere in D dimensions, and $\langle\cdots\rangle$ denotes averaging over u, v and ∇u, ∇v.

We use the following facts: u and v are statistically independent of ∇u and ∇v; u and v are independent of each other and have the same distribution; ∇u and ∇v are independent

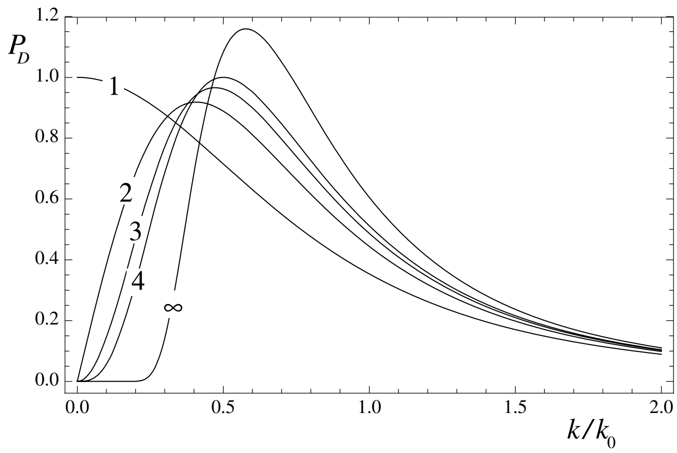

Figure 3. Probability distributions $P_D(k)$ (equation (2.7)) for the indicated values of D.

of each other and have the same distribution; and for isotropic randomness, the components $\partial_i u$ and $\partial_j u$ are independent of each other for $i \neq j$ and have the same distribution. Thus, evaluating the averages over the gradients first, and in an obvious notation,

$$\left\langle \exp\left\{ \mathrm{i}\boldsymbol{s}\cdot\frac{u\nabla v - v\nabla u}{\rho^2}\right\}\right\rangle = \left\langle \exp\left\{-\frac{1}{2\rho^2}\langle(\boldsymbol{s}\cdot(\cos\chi\nabla v - \sin\chi\nabla u))^2\rangle_{\nabla u,\nabla v}\right\}\right\rangle_{\rho,\chi}$$
$$= \left\langle \exp\left\{-\frac{1}{2\rho^2}s^2\langle(\partial_x u)^2\rangle\right\}\right\rangle_{\rho,\chi}. \qquad (2.4)$$

From isotropy and monochromaticity follows

$$\langle(\partial_x u)^2\rangle = \frac{1}{D}\langle|\nabla u|^2\rangle = \frac{k_0^2}{D}\langle u^2\rangle, \qquad (2.5)$$

and so, performing the average over u and v by integrating over ρ and χ,

$$P_D(k) = \frac{\Omega_D k^{D-1} D^{D/2}}{(2\pi)^{D/2} k_0^D \langle u^2\rangle^{D/2}}\left\langle \rho^D \exp\left\{-\frac{Dk^2\rho^2}{2k_0^2\langle u^2\rangle}\right\}\right\rangle_{\rho,\chi}$$
$$= \frac{\Omega_D k^{D-1} D^{D/2}}{(2\pi)^{D/2} k_0^D \langle u^2\rangle^{D/2+1}}\int_0^\infty \mathrm{d}\rho\,\rho^{D+1}\exp\left\{-\frac{\rho^2}{2\langle u^2\rangle}\left(\frac{Dk^2}{k_0^2}+1\right)\right\}. \qquad (2.6)$$

Thus

$$P_D(k) = \frac{k_0^2 k^{D-1}}{\left(k^2 + k_0^2/D\right)^{\frac{1}{2}D+1}}. \qquad (2.7)$$

Several of these distributions are shown in figure 3. In the limit $D \to \infty$,

$$P_\infty(k) = \frac{k_0^2}{k^3}\exp\left\{-\frac{k_0^2}{2k^2 D}\right\}. \qquad (2.8)$$

Now (2.1) gives our main result: the superoscillation probability

$$P_{D\,\mathrm{super}} = 1 - \left(\frac{D}{D+1}\right)^{D/2}. \qquad (2.9)$$

These numbers slowly increase with D; a few values are

$$P_{1\,\text{super}} = 1 - \frac{1}{\sqrt{2}} = 0.292\,89, \qquad P_{2\,\text{super}} = \frac{1}{3},$$
$$P_{3\,\text{super}} = 1 - \frac{3\sqrt{3}}{8} = 0.350\,48, \ldots, \qquad P_{\infty\,\text{super}} = 1 - \frac{1}{\sqrt{\mathrm{e}}} = 0.393\,47. \tag{2.10}$$

The distributions (2.7) decay as k^{-3} for large k, so moments of order 2 or higher diverge. The first moment—the mean value of $k(\boldsymbol{r})$—is

$$\langle k \rangle_D = \int_0^\infty \mathrm{d}k\, k P_D(k) = \sqrt{\frac{\pi}{D}} \frac{\Gamma\left(\frac{1}{2}(1+D)\right)}{\Gamma\left(\frac{1}{2}D\right)}, \tag{2.11}$$

and increases from 1 to $\sqrt{\pi/2}$ as D increases from 1 to infinity.

3. Superoscillations for $D = 1$

For this section only, it is convenient to define the local wavenumber $k(x)$ without the modulus sign in (1.3), so that it can take positive as well as negative values, that is

$$k(x) = \partial_x \chi(x) = \mathrm{Im}[\partial_x \log \psi(x)]. \tag{3.1}$$

The formulae in the previous section have sensible limits when $D = 1$. This might seem strange, because the plane-wave components of a monochromatic wave $\psi(x)$ are concentrated at $k_n = +k_0$ and $k_n = -k_0$, and it is hard to see how such a function can superoscillate. It would seem that the local wavenumber $k(x)$ should be $+k_0$ if the positive amplitude dominates, or $-k_0$ if the negative amplitude dominates. Nevertheless, monochromatic waves in one dimension can superoscillate, as we show now.

It is necessary to take the limit of a nonmonochromatic wave as its spectrum is concentrated onto wavenumbers $+k_0$ and $-k_0$, that is

$$\begin{aligned} &\psi(x) = \sum_{n=1}^{N} a_n \exp\{\mathrm{i}k_n x\}, \\ &-k_0 \leqslant k_n \leqslant -k_0(1-\delta) \qquad \text{for} \quad 1 \leqslant n \leqslant \tfrac{1}{2}N, \\ &k_0(1-\delta) \leqslant k_n \leqslant k_0 \qquad \text{for} \quad \tfrac{1}{2}N + 1 \leqslant n \leqslant N, \quad N \gg 1, \quad \delta \ll 1. \end{aligned} \tag{3.2}$$

As numerical simulation indicates (figure 4), the distribution of local wavenumbers matches the theoretical distribution very well. And the superoscillation fractions for superpositions of $N = 50$ waves, with 25 wavenumbers randomly distributed between $-k_0$ and $-0.99k_0$ and 25 randomly distributed between $0.99k_0$ and k_0 (i.e. $\delta = 0.01$ in (3.1)), typically agree with the theoretical value $P_{1\,\text{super}} = 0.292\,89$ to better than 1% in runs of 10^5 samples.

A sample wave of this type, together with the corresponding local wavenumber $k(x)$, is shown in figure 5. It is clear that the large values of $k(x)$ occur close to the places where $|\psi(x)|$ is small—places that in more dimensions would be the phase singularities.

To understand how this generates the asymptotic superoscillatory distribution $P_\infty(k) \to k_0^2/k^3$ for $k \gg k_0$ (cf (2.7)), it will suffice to study the simple two-wave model,

$$\psi(x) = (1+\varepsilon)\exp\{\mathrm{i}x\} + \exp\{-\mathrm{i}x\}, \tag{3.3}$$

where $\varepsilon \ll 1$ is a real number (any change in the phase of the amplitude can be accommodated by shifting x). From (3.1), the local wavenumber is

$$k(x) = \frac{2\varepsilon + \varepsilon^2}{\varepsilon^2 + 4\cos^2 x}. \tag{3.4}$$

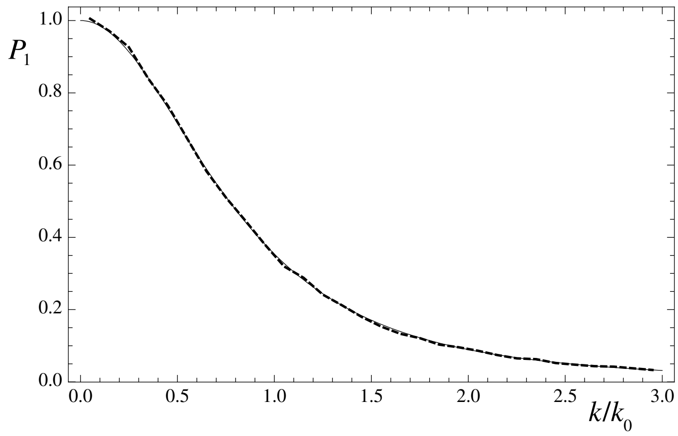

Figure 4. Dashed curve: probability distribution $P_1(k)$, from simulation of 10^5 samples of the wave (3.2) with $N = 50$, $\delta = 0.01$; thin curve (barely visible behind): theoretical distribution (2.7).

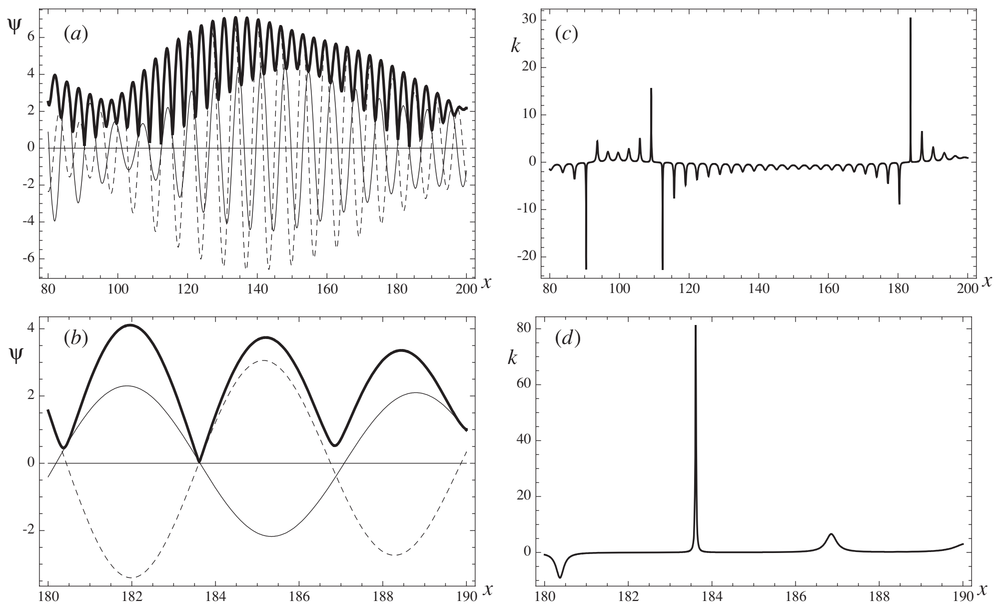

Figure 5. Sample wave (3.2) for $D = 1$, with $N = 50$, $\delta = 0.1$, $k_0 = 1$. (*a*) Thin curve, Re $\psi(x)$; dashed curve, Im $\psi(x)$; thick curve, $|\psi(x)|$. (*b*) Magnification of (*a*). (*c*) Local wavenumber $k(x)$ corresponding to (*a*). (*d*) Magnification of (*c*).

As figure 6 illustrates (for a negative value of ε), the maxima of $k(x)$, of magnitude $2/\varepsilon$, coincide with the minima of $|\psi(x)|$, of magnitude $|\varepsilon|$, at $x = \pi/2 \pmod{\pi}$.

We are interested in large values of $k(x)$, for which

$$k\left(\frac{1}{2} + \xi\right) \approx \frac{2}{\varepsilon} - \frac{8\xi^2}{\varepsilon^3}. \tag{3.5}$$

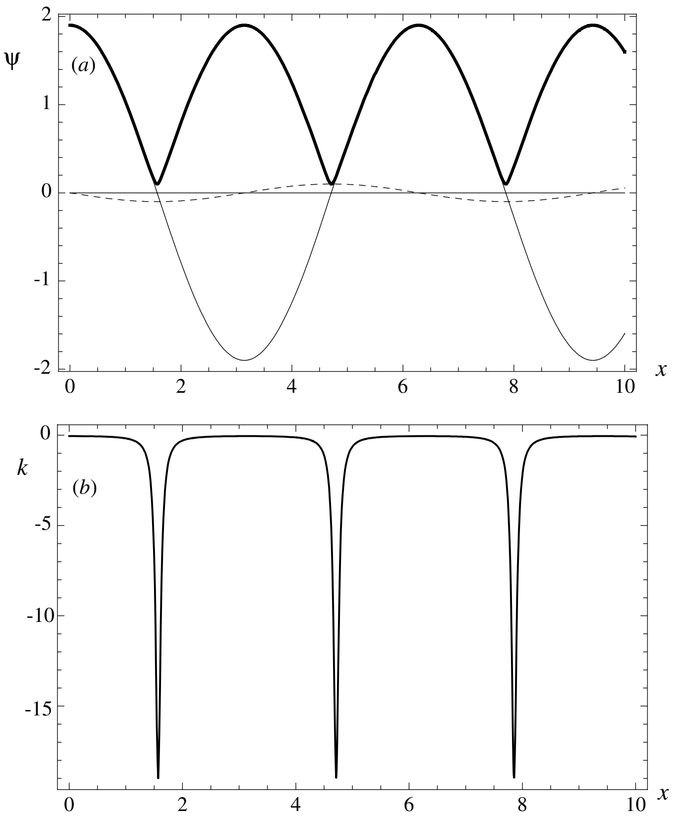

Figure 6. (*a*) Two-wave superposition (3.3) with $\varepsilon = -0.1$: thin curve, $\mathrm{Re}\,\psi(x)$; dashed curve, $\mathrm{Im}\,\psi(x)$; thick curve, $|\psi(x)|$. (*b*) Local wavenumber $k(x)$ (3.4) corresponding to (*a*).

For fixed ε, this basic narrow-parabola model gives the probability distribution of the local wavenumber as

$$P(\varepsilon, k) = \frac{2}{\varepsilon}\int_0^{\varepsilon/2} \mathrm{d}\,\xi\,\delta\left(k - \left(\frac{2}{\varepsilon} - \frac{8\xi^2}{\varepsilon^3}\right)\right) = \frac{\varepsilon}{4\sqrt{1-\frac{1}{2}k\varepsilon}}\Theta\left(k - \frac{2}{\varepsilon}\right). \quad (3.6)$$

In the Gaussian random waves ψ we are interested in, the minimum values ε of $|\psi|$ are not fixed but fluctuate. Therefore, we can estimate $P_1(k)$ by averaging over ε, using the fact that the distribution of minimum values vanishes linearly as $\varepsilon \to 0$. Thus

$$\begin{aligned} P_1(k) &\approx \frac{1}{2}\int_0^2 \mathrm{d}\varepsilon\,\varepsilon\,P(\varepsilon, k) = \frac{2}{15k^3}\left(8 - \Theta\,(1-k)\,\sqrt{1-k}(8+4k+3k^2)\right) \\ &= \frac{16}{15k^3} \qquad \text{if} \quad k > 1, \end{aligned} \quad (3.7)$$

thereby explaining the superoscillations for $D = 1$.

4. Discussion

We have shown that typical waves superoscillate over surprisingly large regions of D-dimensional space, increasing from a fraction 0.293 for $D = 1$ to 0.394 as $D \to \infty$. This reinforces and extends the result $1/3$ recently obtained [8] for $D = 2$. In comparison with the extreme superoscillations that can be artificially constructed [1], those considered here are relatively modest: typically a single large spike in $k(\boldsymbol{r})$ as $\boldsymbol{r}$ passes a nearby phase singularity. Nevertheless, the probability distribution of $k(\boldsymbol{r})$ has a long superoscillatory tail, slowly decaying as k^{-3}.

Beyond statistics, much remains to be explored about the geometry and topology of superoscillatory regions. Preliminary evidence for $D = 2$ suggests that these regions are usually connected, including infinitely many wave vortices in percolating clusters. The same could be true for $D > 2$, but we do not know.

We have concentrated on the superoscillations, for which $k(\boldsymbol{r}) > k_0$, and the related large-k asymptotics of $P_D(k)$. However, we note that for $k \to 0$, that is suboscillations (phase variations much slower than the common wavelength λ), $P_D(k)$ vanishes as k^{D-1} (equation (2.7)), and is nonanalytic for $D = \infty$ (equation (2.8)).

Wave superoscillations have quantum implications that will be explored elsewhere. In brief, large values of $k(\boldsymbol{r})$ could be measured in a quantum weak measurement [9], as high-momentum impulses from quanta (photons in the case of optical fields) detected by detectors small compared with the wavelength, for example narrow slits [10]. Momentum arises because $k(\boldsymbol{r})$ can be regarded as the local expectation value of the momentum operator. In an obvious quantum notation,

$$\boldsymbol{k}(\boldsymbol{r}) = \langle\psi|\hat{\boldsymbol{p}}(\boldsymbol{r})|\psi\rangle, \tag{4.1}$$

where

$$\hat{\boldsymbol{p}}(\boldsymbol{r}) = \tfrac{1}{2}(\delta(\hat{\boldsymbol{r}} - \boldsymbol{r})\hat{\boldsymbol{p}} + \hat{\boldsymbol{p}}\delta(\hat{\boldsymbol{r}} - \boldsymbol{r})), \tag{4.2}$$

and, in position representation,

$$\hat{\boldsymbol{p}} = -\mathrm{i}\nabla. \tag{4.3}$$

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